Appendix C

Some Summation Formulas

Let $A(m, n)$ be a function of two discrete variables $m, n = 0, 1, 2, \ldots$, i.e., on the points of Figure C.1. We assume that the sum S converges, regardless of the order in which it is performed:

$$S := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A(m, n) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A(m, n). \tag{C.1}$$

This only means that in one case we sum over columns and in the other over rows. We can also perform the summation over diagonals (thin lines in Fig. C.1) as

$$S = [A(0, 0)] + [A(0, 1) + A(1, 0)] + [A(0, 2) + A(1, 1) + A(2, 0)] + \cdots + \left[\sum_{m=0}^{n} A(m, n - m)\right] + \cdots = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A(m, n - m). \tag{C.2}$$

The same sum can be done following 60° directions (broken lines):

$$S = [A(0, 0)] + [A(0, 1)] + [A(0, 2) + A(1, 0)] + [A(0, 3) + A(1, 1)] + \cdots + \left[\sum_{m=0}^{n/2} A(m, n - 2m)\right]_{n \text{ even}} + \left[\sum_{m=0}^{(n-1)/2} A(m, n - 2m)\right]_{n \text{ odd}} + \cdots = \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} A(m, n - 2m), \tag{C.3}$$

where $[n/2]$ is the largest integer not exceeding $n/2$. For example, $[2] = 2$, $[5/2] = [2.5] = 2$, etc. The same argument can be used to sum the $A(m, n)$ along lines which run one unit to the right and r units down, obtaining

$$S = \sum_{n=0}^{\infty} \sum_{m=0}^{[n/r]} A(m, n - rm). \tag{C.4}$$

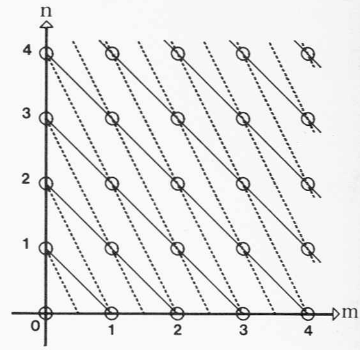

Fig. C.1. Various summation orders on an infinite two-dimensional lattice of non-negative integers.

When triple sums of $A(k, m, n)$ appear, the sum can be performed over diagonals in space; thus

$$\begin{aligned} T &:= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A(k, m, n) \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{min(k,m)} A(k - n, m - n, n) \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{min([k/r],[m/s])} A(k - rn, m - sn, n). \end{aligned} \tag{C.5}$$

The middle term has been used in (7.183). Equations (C.1)–(C.4) can now be employed to produce further identities involving the first two indices of the sum.